Policy change and DSGE models

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Outline

1. DSGE models, policy and structural change
2. Structural change as deterministic exogenous variables
3. Linear approximation with structural change
4. Second order approximation with structural change
5. Application: tax rate change in a monopolistic competition model
6. Further developments

Structural change

- Most existing DSGE models have constant parameters and zero-mean shocks.
- Can’t handle long run tendencies such as demographic change or policy shift.
- Expected change triggers anticipatory behaviour.
- Unexpected change only in a radical sense: its occurrence isn’t even thought possible. Otherwise it must be modeled as a stochastic shock.
- Usually discussed only in deterministic models.

Formal treatment of structural change

- Representing structural change as deterministic exogenous variables
- Unexpected change simply shift the simulation but doesn't enter expectations
- Expected change is part of information set when decisions are taken

Proposed solution: add future values of deterministic exogenous variables to the list of state variables.

A general model

A DSGE model can be represented by a set of stochastic equations:

$$E_t \left\{ f(y_{t+1}, y_t, y_{t-1}, x_t, u_t) \right\} = 0$$

where y_t is the vector of endogenous variables in the model, x_t is a vector of exogenous deterministic variables. By assumption, these variables settle to a constant value after some horizon, $x_\tau = \bar{x}$ for $\tau > T_N$. u_t is a vector of stochastic shocks defined as $u_t = \sigma \epsilon_t$, whith

$$E\{\epsilon_t\} = 0 \qquad\qquad E\{\epsilon_t \epsilon_t^T\} = \Sigma_\epsilon.$$

σ is a stochastic scale factor.

Perturbation approach

Unknown decision functions:

$$y_t = g(y_{t-1}, x_t, \dots, x_{t+N}, u_t, \sigma)$$

Recover the Taylor expansion coefficients of the unknown decision functions $g(\dots)$ from the Taylor expansion of the structural model $E\left\{f(\dots)\right\} = 0$.
WARNING: the perturbation approach involves a local approximation that may not be satisfactory for large changes in exogenous variables

Deterministic steady state

$$
\begin{aligned}
&(1) \qquad f(\bar{y}, \bar{y}, \bar{y}, \bar{x}, 0) &= 0 \\
&(2) \qquad \bar{y} &= g(\bar{y}, \bar{x}, \ldots, \bar{x}, 0, 0)
\end{aligned}
$$

Note that the deterministic steady state is computed at the value at which the exogenous variables finally settle.

The model as function of state variables

$$
\begin{aligned}
y_t &= g(y_{t-1}, x_t, \ldots, x_{t+N}, u_t, \sigma) \\
y_{t+1} &= g(y_t, x_{t+1}, \ldots, x_{t+N}, \bar{x}, u_{t+1}, \sigma) \\
&= g(g(y_{t-1}, x_t, \ldots, x_{t+N}, u_t, \sigma), x_{t+1}, \ldots, x_{t+N}, \bar{x}, u_{t+1}, \sigma)
\end{aligned}
$$

$$
\begin{aligned}
&F(y_{t-1}, x_t, \ldots, x_{t+N}, u_t, u_{t+1}, \sigma) = \\
&\qquad f(g(g(y_{t-1}, x_t, \ldots, x_{t+N}, u_t, \sigma), x_{t+1}, \ldots, x_{t+N}, \bar{x}, u_{t+1}, \sigma), \\
&\qquad\quad g(y_{t-1}, x_t, \ldots, x_{t+N}, u_t, \sigma), y_{t-1}, x_t, u_t, \sigma)
\end{aligned}
$$

$$E_t\left\{F(y_{t-1}, x_t, \ldots, x_{t+N}, u_t, u_{t+1}, \sigma)\right\} = 0$$

First order approximation

$$F^{(1)}(y_{t-1},x_t,\ldots,x_{t+N},u_t,u_{t+1},\sigma) =$$
$$F(\bar{y},\bar{x},\ldots,\bar{x},0,0)+F_y\hat{y}+F_{x_1}\hat{x}_1+\ldots+F_{x+N}\hat{x}_N+F_u u+F_{u'}u'+F_\sigma\sigma$$

with $\hat{y}=y_{t-1}-\bar{y}$, $\hat{x}_1=x_t-\bar{x}$, $\ldots$, $\hat{x}_N=x_{t+N}-\bar{x}$, $u=u_t$, $u'=u_{t+1}$.

$$\begin{aligned}
E_t\left\{F^{(1)}(y_{t-1},\hat{x}_t,\ldots,\hat{x}_{t+N},u_t,u_{t+1},\sigma)\right\} \\
&= F(\bar{y},\bar{x},\ldots,\bar{x},0,0)+F_y\hat{y}+F_{x_1}\hat{x}_1+\ldots+F_{x+N}\hat{x}_N+F_u u+F_\sigma\sigma \\
&= 0 \\
F_y &= 0 \\
F_u &= 0 \\
F_\sigma &= 0 \\
F_{x_1} &= 0 \\
\ldots \\
F_{x_N} &= 0
\end{aligned}$$

Recovering g_y, g_u, g_σ

- g_y is recovered from $F_y = 0$
- g_u is recovered from $F_u = 0$
- $g_\sigma (= 0)$ is recovered from $F_\sigma = 0$

g_x

$$\begin{aligned}
F_{x_1} &= f_+ g_y g_{x_1} + f_0 g_{x_1} + f_x \\
&= 0 \\
g_{x_1} &= -\left(f_+ g_y + f_0\right)^{-1} f_x \\
F_{x_i} &= f_+ \left(g_y g_{x_i} + g_{x_{i-1}}\right) + f_0 g_{x_i} && i = 2, \dots, N \\
&= 0 \\
g_{x_i} &= -\left(f_+ g_y + f_0\right)^{-1} f_+ g_{x_{i-1}} && i = 2, \dots, N
\end{aligned}$$

Second order approximation

$$
\begin{aligned}
&F^{(2)}(y_{t-1},x_t,\ldots,x_{t+N},u_t,u_{t+1},\sigma) = \\
&\qquad F^{(1)}(y_{t-1},x_t,\ldots,x_{t+N},u_t,u_{t+1},\sigma) + 0.5(F_{yy}(\hat{y}\otimes\hat{y}) + F_{uu}(u\otimes u) + F_{u'u'}(u'\otimes u') \\
&\qquad +F_{\sigma\sigma}\sigma^2) + F_{yu}(\hat{y}\otimes u) + F_{yu'}(\hat{y}\otimes u') + F_{yx_1}(\hat{y}\otimes\hat{x}_1) + \ldots + F_{yx_N}(\hat{y}\otimes\hat{x}_N) \\
&\qquad +F_{y\sigma}(\hat{y}\otimes\sigma) + F_{uu'}(u\otimes u') + F_{ux_1}(u\otimes\hat{x}_1) + \ldots + F_{ux_N}(u\otimes\hat{x}_N) + F_{u\sigma}(u\otimes\sigma) \\
&\qquad +F_{u'x_1}(u'\otimes\hat{x}_1) + \ldots + F_{u'x_N}(u'\otimes\hat{x}_N) + F_{u'\sigma}(u'\otimes\sigma) + F_{x_1x_2}(\hat{x}_1\otimes\hat{x}_2) \\
&\qquad +\ldots + F_{x_{N-1}x_N}(\hat{x}_{N-1}\otimes\hat{x}_N) + F_{x_1\sigma}(\hat{x}_1\sigma) + \ldots + F_{x_N\sigma}(\hat{x}_N\sigma) \\
&E_t\left\{F^{(2)}(y_{t-1},x_t,\ldots,x_{t+N},u_t,u_{t+1},\sigma)\right\} = \\
&\qquad F^{(1)}(y_{t-1},x_t,\ldots,x_{t+N},u_t,u_{t+1},\sigma) + 0.5(F_{yy}(\hat{y}\otimes\hat{y}) + F_{uu}(u\otimes u) + F_{u'u'}\sigma^2\vec{\Sigma}_\epsilon \\
&\qquad +F_{\sigma\sigma}\sigma^2) + F_{yu}(\hat{y}\otimes u) + F_{yx_1}(\hat{y}\otimes\hat{x}_1) + \ldots + F_{yx_N}(\hat{y}\otimes\hat{x}_N) + F_{y\sigma}(\hat{y}\otimes\sigma) \\
&\qquad +F_{ux_1}(u\otimes\hat{x}_1) + \ldots + F_{ux_N}(u\otimes\hat{x}_N) + F_{u\sigma}(u\otimes\sigma) + F_{x_1x_2}(\hat{x}_1\otimes\hat{x}_2) \\
&\qquad +\ldots + F_{x_{N-1}x_N}(\hat{x}_{N-1}\otimes\hat{x}_N) + F_{x_1\sigma}(\hat{x}_1\sigma) + \ldots + F_{x_N\sigma}(\hat{x}_N\sigma) \\
&\quad = \quad 0
\end{aligned}
$$

Recovering g_{yy}, g_{yu}, g_{uu}

- g_{yy} is recovered from $F_{yy} = 0$
- g_{yu} is recovered from $F_{yu} = 0$
- g_{uu} is recovered from $F_{uu} = 0$

Recovering $g_{\sigma\sigma}$

$$\begin{aligned} F_{u'u'}\sigma^2\vec{\Sigma}_\epsilon + F_{\sigma\sigma}\sigma^2 &= 0 \\ \left(f_{++}(g_u \otimes g_u) + f_+ g_{uu}\right)\vec{\Sigma}_\epsilon + f_+\left(g_{\sigma\sigma} + g_y g_{\sigma\sigma}\right) + f_0 g_{\sigma\sigma} &= 0 \end{aligned}$$

$$g_{\sigma\sigma} = -\left(f_+(I + g_y) + f_0\right)^{-1}\left(f_{++}(g_u \otimes g_u) + f_+ g_{uu}\right)\vec{\Sigma}_\epsilon$$

and

$$g_{y\sigma} = g_{u\sigma} = g_{x_1\sigma} = \ldots = g_{x_N\sigma} = 0$$

Recovering g_{yx_i}, g_{ux_i}

$$\begin{aligned}
F_{yx_1} &= f_+ g_y g_{yx_1} + f_0 g_{yx_1} + R_1 \\
&= 0 \\
g_{yx_1} &= -\left(f_+ g_y + f_0\right)^{-1} R_1 \\
F_{yx_i} &= f_+ \left(g_{yx_{i-1}}(g_y \otimes I) + g_y g_{yx_i}\right) + f_0 g_{yx_i} + R_i \\
&= 0 \\
g_{yx_i} &= -\left(f_+ g_y + f_0\right)^{-1} \left(f_+ g_{yx_{i-1}}(g_y \otimes I) + R_i\right) \\
& \quad i = 2, \ldots, N
\end{aligned}$$

where the terms $R_1, \ldots, R_N$, don't contain second order derivatives of $g()$ with respect to x_1,.

g_{ux_i} is recovered in a similar manner from $F_{ux_i} = 0$.

Recovering $g_{x_i x_j}$

$$\begin{aligned}
F_{x_1 x_1} &= f_+ g_y g_{x_1 x_1} + f_0 g_{x_1 x_1} + R_{11} \\
&= 0 \\
g_{x_1 x_1} &= -\left(f_+ g_y + f_0\right)^{-1} R_{11} \\
F_{x_1 x_i} &= f_+ \left(g_{y x_{i-1}}(g_{x_1} \otimes I) + g_y g_{x_1 x_i}\right) + f_0 g_{x_1 x_i} + R_{1i} \\
&= 0 \\
g_{x_1 x_i} &= -\left(f_+ g_y + f_0\right)^{-1} \left(f_+ g_{y x_{i-1}}(g_{x_1} \otimes I) + R_{1i}\right) \\
F_{x_i x_j} &= f_+ \left(g_{y x_{j-1}}(g_{x_i} \otimes I) + g_{x_{i-1} y}(I \otimes g_{x_j}) + g_y g_{x_i x_j} + g_{x_{i-1} x_{j-1}}\right) + f_0 g_{x_i x_j} + R_{ij} \\
&= 0 \\
g_{x_i x_j} &= -\left(f_+ g_y + f_0\right)^{-1} \left(f_+ \left(g_{y x_{j-1}}(g_{x_i} \otimes I) + g_{x_{i-1} y}(I \otimes g_{x_j}) + g_{x_{i-1} x_{j-1}}\right) + R_{ij}\right) \\
& \quad i = 2, \ldots, N \qquad j = 2, \ldots, N
\end{aligned}$$

where the terms $R_{11}, \ldots, R_{NN}$, don’t contain second order derivatives of $g()$ with respect to $x_1, \ldots, x_N$.

Approximated decision functions

$$\begin{aligned}
y_t \approx\; & \bar{y} + 0.5 g_{\sigma\sigma} + g_y \hat{y} + g_u \hat{u} + g_{x_1} \hat{x}_1 + \ldots + g_{x_N} \hat{x}_N + 0.5 (g_{yy} (\hat{y} \otimes \hat{y}) \\
& + g_{uu} (u \otimes u) + g_{x_1 x_1} (\hat{x}_1 \otimes \hat{x}_1) + \ldots + g_{x_N x_N} (\hat{x}_N \otimes \hat{x}_N)) + g_{yu} (\hat{y} \otimes u) \\
& + g_{y x_1} (\hat{y} \otimes x_1) + \ldots + g_{y x_N} (\hat{y} \otimes x_N) + g_{u x_1} (u \otimes x_1) + \ldots + g_{u x_N} (u \otimes x_N) \\
& + g_{x_1 x_2} (\hat{x}_1 \otimes x_2) + \ldots + g_{x_{N-1} x_N} (\hat{x}_{N-1} \otimes x_N)
\end{aligned}$$

Hairault, Langot and Portier (2001) model

Welfare:

$$W_t = \ln c_t + \eta \ln(1 - h_t) + \beta E_t \left\{ W_{t+1} \right\}$$

with W, welfare, c, consumption, and h, labor effort. $\eta = 2$

Optimality condition for consumption

$$\frac{1}{c_t} = E_t \left\{ \beta \frac{1}{c_{t+1}} (z_{t+1} + 1 - \delta) \right\}$$

where z is the rate of return on capital, net of taxes.
$\beta = 0.988$, $\delta = 0.025$

Model (continued)

Optimality condition for labor effort

$$\frac{\eta}{1-h_t} = \frac{\omega_t}{c_t}$$

where ω is the wage rate, net of taxes.

Net wage rate

$$(1-\alpha)\left(\frac{k_{t-1}}{h_t}\right)^{\alpha} = (1+\mu)(1+\tau_t)\omega_t$$

where k_{t-1} is the stock of capital at the end of the previous period and τ is the tax rate. $\alpha = 0.36$, $\mu = 0.1$

Model (continued)

Net rate of return on capital

$$\alpha \left(\frac{k_{t-1}}{h_t} \right)^{\alpha - 1} = (1 + \mu)(1 + \tau_t) z_t$$

Accumulation

$$i_t = k_t - (1 - \delta) k_{t-1}$$

with i, investment. $\delta = 0.025$

Model (continued)

Goods market equilibium

$$c_t + i_t = A_t k_{t-1}^{\alpha} h_t^{1-\alpha}$$

Total factor productivity

$$\ln A_t = (1-\rho)\ln \bar{A} + \rho \ln A_{t-1} + e_t$$

where $\rho = 0.95$.

A surprise change in the tax rate

1. The economy is at the deterministic steady state corresponding to a tax rate $\tau = -0.15$ (arbitrary initial state)
2. The tax rate is suddenly moved to the optimal value $\tau = -\frac{\mu}{1+\mu} \approx -0.0909$.

Results

For the case where the value of 0 would be drawn for the TFP shock:

Variable	Period 0	Period 1	
		$\tau = -0.15$	$\tau = -\frac{\mu}{1+\mu}$
c	0.8723	0.8593	0.8769
h	0.3160	0.3217	0.2889
i	0.3050	0.3319	0.2332
ω	2.5506	2.5338	2.4644
z	0.0371	0.0376	0.0329
W	NA	-75.8701	-75.3735

The welfare measure corresponds to the ranking of policies after the shock of period 1 is observed.

A pre–announced change in tax rate

Same experiment as before, but the change in policy is announced 10 periods before taking place.

We compute the expected path of the variables, conditional on the initial starting point (deterministic steady state for $\tau = -0.15$)

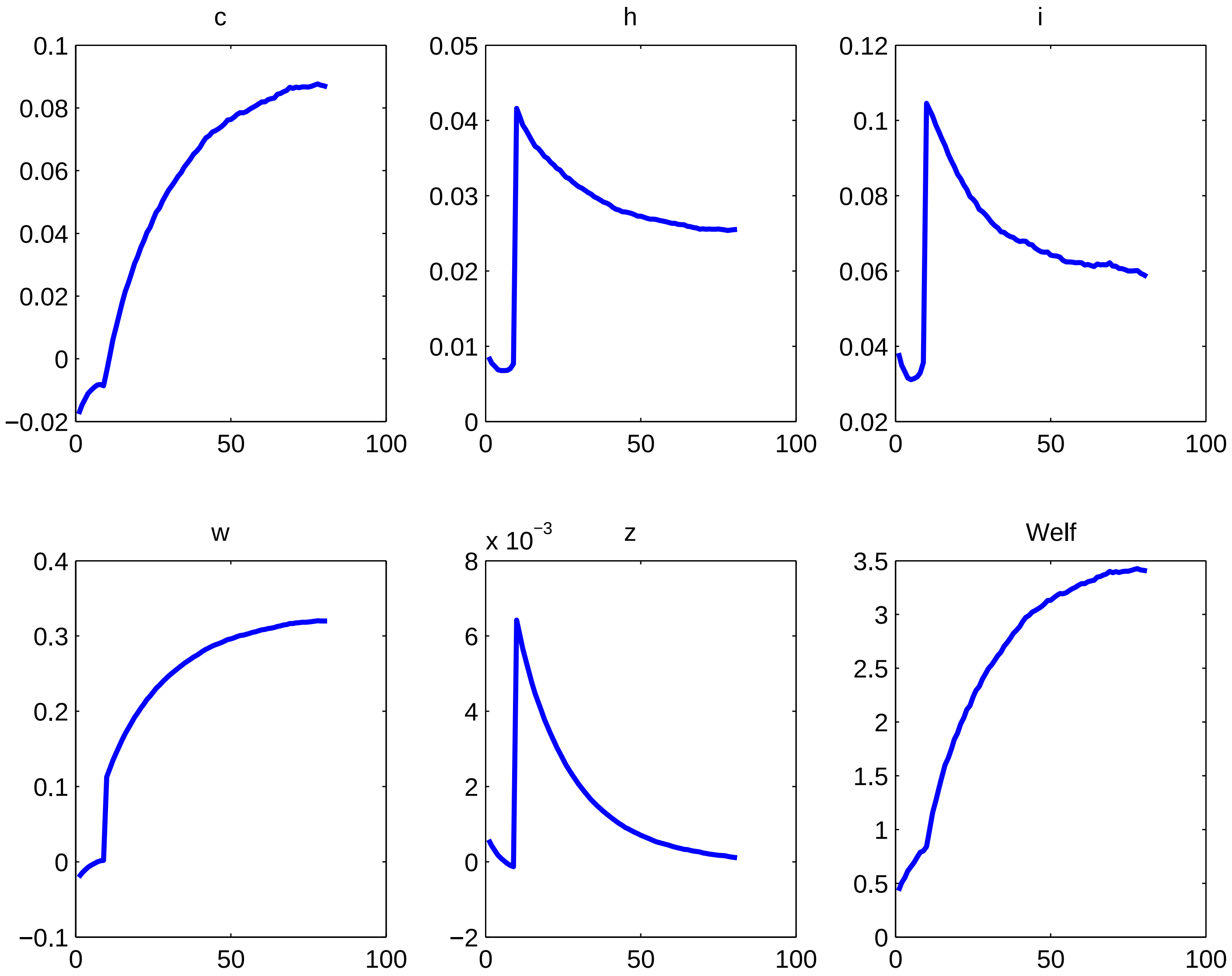
This time, the welfare criterium is conditional on period 0, rather than period 1

$$E_0\left\{\sum_{t=1}^{\infty}\beta^{t-1}U_t\right\}$$

versus

$$E_1\left\{\sum_{t=1}^{\infty}\beta^{t-1}U_t\right\} = U_1 + E_1\left\{\sum_{t=2}^{\infty}\beta^{t-1}U_t\right\}$$

Results


c
0.1
0.08
0.06
0.04
0.02
0
−0.02
0
50
100
h
0.05
0.04
0.03
0.02
0.01
0
0
50
100
i
0.12
0.1
0.08
0.06
0.04
0.02
0
50
100
w
0.4
0.3
0.2
0.1
0
−0.1
0
50
100
x 10−3
z
8
6
4
2
0
−2
0
50
100
Welf
3.5
3
2.5
2
1.5
1
0.5
0
0
50
100

Accuracy appraisal

With no stochastic shocks it is possible to compare 2nd order approximation with deterministic simulation

Variable	Initial $\tau = -0.25$		Initial $\tau = -0.15$	
	LBJ	O2	LBJ	O2
c	1.0648	1.0582	0.8882	0.8880
h	0.2578	0.2608	0.2836	0.2837
i	0.0912	0.0854	0.2104	0.2102
w	2.8694	2.8484	2.4795	2.4789
z	0.0250	0.0284	0.0324	0.0325
$Welf$	-69.7491	-70.0823	-74.3219	-74.3327

Further developments

- Effects of policy change on volatility can only be captured by 3rd order approximation
- Include expected policy change in estimation
- Uncertainty around future policy change
- Infinite series of deterministic exogenous changes (around which point to approximate?)